

A Hundred-dollar, Hundred-digit Challenge

Each October, a few new graduate students arrive in Oxford to begin research for a doctorate in numerical analysis. In their first term, working in pairs, they take an informal course called the “Problem Solving Squad.” Each week for six weeks, I give them a problem, stated in a sentence or two, whose answer is a single real number. Their mission is to compute that number to as many digits of precision as they can.

Ten of these problems appear below. I would like to offer them as a challenge to the SIAM community. Can you solve them?

I will give \$100 to the individual or team that delivers to me the most accurate set of numerical answers to these problems before May 20, 2002. With your solutions, send in a few sentences or programs or plots so I can tell how you got them. Scoring will be simple: You get a point for each correct digit, up to ten for each problem, so the maximum score is 100 points.

Fine print? You are free to get ideas and advice from friends and literature far and wide, but any team that enters the contest should have no more than half a dozen core members. Contestants must assure me that they have received no help from students at Oxford or anyone else who has already seen these problems.

Hint: They’re hard! If anyone gets 50 digits in total, I will be impressed. The ten magic numbers will be published in the July/August issue of *SIAM News*, together with the names of winners and strong runners-up.—Nick Trefethen, *Oxford University*.

The Hundred-dollar, Hundred-digit Challenge Problems

1. What is $\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 x^{-1} \cos(x^{-1} \log x) dx$?
2. A photon moving at speed 1 in the x - y plane starts at $t = 0$ at $(x, y) = (0.5, 0.1)$ heading due east. Around every integer lattice point (i, j) in the plane, a circular mirror of radius $1/3$ has been erected. How far from the origin is the photon at $t = 10$?
3. The infinite matrix A with entries $a_{11} = 1$, $a_{12} = 1/2$, $a_{21} = 1/3$, $a_{13} = 1/4$, $a_{22} = 1/5$, $a_{31} = 1/6$, etc., is a bounded operator on ℓ^2 . What is $\|A\|$?
4. What is the global minimum of the function

$$\exp(\sin(50x)) + \sin(60e^x) + \sin(70 \sin(x)) + \sin(\sin(80y)) - \sin(10(x+y)) + \frac{1}{4}(x^2 + y^2)$$
?
5. Let $f(z) = 1/\Gamma(z)$, where $\Gamma(z)$ is the gamma function, and let $p(z)$ be the cubic polynomial that best approximates $f(z)$ on the unit disk in the supremum norm $\|\cdot\|_{\infty}$. What is $\|f - p\|_{\infty}$?
6. A flea starts at $(0, 0)$ on the infinite 2D integer lattice and executes a biased random walk: At each step it hops north or south with probability $1/4$, east with probability $1/4 + \varepsilon$, and west with probability $1/4 - \varepsilon$. The probability that the flea returns to $(0, 0)$ sometime during its wanderings is $1/2$. What is ε ?
7. Let A be the $20,000 \times 20,000$ matrix whose entries are zero everywhere except for the primes $2, 3, 5, 7, \dots, 224737$ along the main diagonal and the number 1 in all the positions a_{ij} with $|i - j| = 1, 2, 4, 8, \dots, 16384$. What is the $(1, 1)$ entry of A^{-1} ?
8. A square plate $[-1, 1] \times [-1, 1]$ is at temperature $u = 0$. At time $t = 0$ the temperature is increased to $u = 5$ along one of the four sides, and heat then flows into the plate according to $u_t = \Delta u$. When does the temperature reach $u = 1$ at the center of the plate?
9. The integral $I(\alpha) = \int_0^2 [2 + \sin(10\alpha)] x^{\alpha} \sin(\alpha/(2-x)) dx$ depends on the parameter α . What is the value $\alpha \in [0, 5]$ at which $I(\alpha)$ achieves its maximum?
10. A particle at the center of a 10×1 rectangle undergoes Brownian motion (i.e., 2D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?

Solutions should be sent to Nick Trefethen at Oxford University (LNT@comlab.ox.ac.uk), no later than May 20, 2002.